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Note

A simple a.s. correct algorithm for deciding if a graph has a perfect matching

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Abstract

A fast algorithm for deciding whether a graph of even order has a perfect matching is given. This algorithm has very low probability of error even in the case of graphs with no perfect matching and no isolated points.

1. Introduction

We consider finite simple graphs $G = (V(G), E(G))$. A perfect matching in a graph G with $|V(G)| = n$, $n = 2k$, is a set of k independent edges. We want to consider fast algorithms for deciding whether a graph has a perfect matching (PM) or not, with very low probability of error. A simple first approach is the following:

Algorithm A_0.
- For each vertex $v \in V(G)$ check if v is isolated.
- If there exists an isolated v, output "G has no PM",
  Else output "G has PM".
- Stop.

Since the threshold functions for the properties "G has PM" and "G has no isolated point" are the same, the probability of error for Algorithm A_0 is asymptotically zero (see [2, Ch. VII]).

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The purpose of this work is to propose a simple way to significantly improve on Algorithm A_0.

An isolated point can be considered as a "forbidden structure" as defined in [3] for PM, in the sense that containing an isolated point implies that no PM is present in the graph. To improve on Algorithm A_0 we will show that within those graphs with no isolated points, there exist forbidden structures for PM which are easy to find (if present) and are possessed by almost all graphs with no PM. Then, if we check for both, isolated points and the forbidden structure to be described in a moment, we have a probability of error which is o(probability of error of A_0) at very little extra computational cost.

2. A simple algorithm for deciding if a graph has a perfect matching with very low probability of error

Let us begin by quoting a classic result of Tutte [4].

Theorem 2.1. *A nontrivial graph G has a perfect matching if and only if for every $S \subset V(G)$, the number of odd connected components of $G - S$ does not exceed $|S|$.*

Tutte's theorem gives a method for deciding PM which has the disadvantage of being computationally intensive. Fortunately, one can get away (almost surely) by checking only for the forbidden structures of very small size among those given by Theorem 2.1.

Let $S \subset V(G)$. If $G - S$ has more than $|S|$ odd connected components, we call S a forbidden structure for PM of type $(|S|; \gamma_1, \gamma_2, \ldots, \gamma_{|S|+1})$, where $\gamma_1, \gamma_2, \ldots, \gamma_{|S|+1}$ are the cardinalities, in increasing order, of the smallest $|S| + 1$ odd components in $G - S$. If an isolated point is present in G, then the empty set is a forbidden structure of type $(0; 1)$. We know that if G has no PM then almost surely a forbidden structure of type $(0; 1)$ is present in the graph, and this is the point in using Algorithm A_0. Another forbidden structure for PM consists in a vertex v such that $G - v$ contains two isolated points, that is, a forbidden structure of type $(1; 1, 1)$. We have the following

Theorem 2.2. *Let $q_1 = \Pr\{G$ contains forbidden structure of type $(1; 1, 1)\}$, and $q_2 = \Pr\{G$ contains forbidden structure for PM of type different from $(0; 1)$ and $(1; 1, 1)\}$. Then $q_2 = o(q_1)$.*

All probabilities in Theorem 2.2 are with respect to the uniform distribution in the set of graphs with n vertices.

In view of this result, let us propose the following algorithm for deciding whether G has a perfect matching.

Algorithm A_1.

- For each vertex $v \in V(G)$ check if v is isolated.
- If there exists an isolated v, output "G has no PM" and stop.

Else
- For each $v \in V(G)$ check if the graph $G - v$ contains two isolated points
- If such a v exists output "G has no PM" and stop.

- Endif
- Output "G has PM" and stop.

Theorem 2.3. *Let G be randomly chosen according to the uniform distribution on the set of all graphs on $n = 2k$ vertices. Let p_0 and p_1 be the probabilities of error using Algorithm* A_0 *and Algorithm* A_1, *respectively. Then $p_1 = \mathrm{o}(p_0)$.*

A naive implementation of Algorithm A_0 has computational complexity $\mathrm{O}(n^2)$. In the same manner, we get complexity $\mathrm{O}(n^3)$ for Algorithm A_1. Theorem 2.3 says that this extra computational cost is justified by a substantial reduction in the probability of error.

3. Proof of results

Theorem 2.3 is a direct consequence of Theorem 2.2, therefore we will only discuss the proof of the latter. Firstly, we introduce some definitions and a lemma.

A partition of a positive integer n is a finite nondecreasing sequence of positive integers $\lambda = \lambda_1, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition. Let $\Pi(n, M) = \{\lambda\colon \lambda$ is a partition of n into exactly M parts$\}$, $\Pi_0(n, M) = \{\lambda\colon \lambda$ is a partition of n into exactly M odd parts$\}$ and $p_0(n, M) = |\Pi_0(n, M)|$.

For $1 \leqslant s \leqslant n/2$ and $\lambda \in \Pi(n - s, s + 2)$, let

$$A(\lambda) = \binom{n-s}{\lambda_1, \ldots, \lambda_{s+2}} 2^{-\sum_{i \neq j} \lambda_i \lambda_j}.$$

Lemma 3.1. *Let $n_0 = (n - s)/(s + 2)$ and $\lambda \in \Pi(n - s, s + 2)$. Let us suppose that there exist k, l, $1 \leqslant k, l \leqslant s + 2$, such that $\lambda_k > n_0$, $\lambda_l < n_0$ and $\lambda_k - \lambda_l \geqslant 2$. If $\beta \in \Pi(n - s, s + 2)$ satisfies $\beta_i = \lambda_i$ if $i \neq k, l$, $\beta_k = \lambda_k - 1$ and $\beta_l = \lambda_l + 1$, then $A(\lambda) > A(\beta)$.*

Proof.

$$\frac{A(\lambda)}{A(\beta)} = \frac{\lambda_l + 1}{\lambda_k} 2^{\lambda_k - \lambda_l - 1}.$$

Let $r = \lambda_k - \lambda_l$. Then

$$\frac{A(\lambda)}{A(\beta)} = \frac{\lambda_l + 1}{\lambda_l + r} 2^{r-1} = L(r).$$

Since $L'(r) > 0$ for all $r \geqslant 2$, we have

$$\frac{A(\lambda)}{A(\beta)} \geqslant L(2) = 2\frac{\lambda_l + 1}{\lambda_l + 2} > 1. \qquad \square$$

One can easily check that Lemma 3.1 implies $A(\lambda') \geqslant A(\lambda)$, for each $\lambda \in \Pi(n-s, s+2)$, where $\lambda' = \lambda'_1, \ldots, \lambda'_{s+2}$, $\lambda'_1 = \cdots = \lambda'_{s+1} = 1$ and $\lambda'_{s+2} = n - 2s - 1$.

Proof of Theorem 2.2. Let

$$q_{2,0} = \Pr\{G \text{ contains forbidden structure of type } (0;k) \text{ for some odd } k > 1\},$$

$$q_{2,1} = \Pr\{G \text{ contains forbidden structure of type } (1;1,k) \text{ for some odd } k > 1\},$$

$$q^*_{2,1} = \Pr\{G \text{ contains forbidden structure of type } (1;j,k) \text{ for some odd } j,k > 1\},$$

and, for $s \geqslant 2$

$$q_{2,s} = \Pr\{G \text{ contains forbidden structure } S \text{ with cardinality } s\}.$$

Clearly

$$q_2 = q_{2,0} + q_{2,1} + q^*_{2,1} + \sum_{s=2}^{n/2-1} q_{2,s}. \tag{1}$$

For $s \geqslant 2$, bounding the number of connected graphs on λ_i points by the total number of graphs on those points, we have

$$q_{2,s} < \sum_{\lambda \in \Pi_0(n-s,s+2)} \binom{n}{s} \binom{n-s}{\lambda_1, \ldots, \lambda_{s+2}} 2^{\binom{s}{2} + \sum_i \binom{\lambda_i}{2} + s(n-s) - \binom{n}{2}}.$$

It follows that

$$q_{2,s} < \sum_{\lambda \in \Pi_0(n-s,s+2)} \binom{n}{s} A(\lambda) < p_0(n-s, s+2) \binom{n}{s} A(\lambda').$$

It is easy to see that $p_0(n-s, s+2) \leqslant \binom{n/2}{s+1}$, then

$$q_{2,s} < (s+1)! \binom{n/2}{s+1} \binom{n}{s} \binom{n-s}{n-2s-1} 2^{-\frac{s(s+1)}{2} - (s+1)(n-2s-1)}.$$

Now for $s \geqslant 2$, let $C(s) = q_{2,s}/q_1$. Using that $q_1 > \frac{1}{2}\binom{n}{3} 2^{5-2n}$ and Stirling's approximation, we get

$$C(s) < \frac{12 n^{2s-2} e^{2s} (n-s)^{s+1} 2^{(3/2)s^2 + (3/2)s - ns + n - 5}}{(s+1) s^{2s+1}}.$$

A lengthy, but straightforward calculation shows that for $n \geqslant 3$, $C(s) < C(2)$ for each $2 \leqslant s \leqslant n/2 - 1$, then we have

$$\frac{1}{q_1} \sum_{s=2}^{n/2-1} q_{2,s} < (n/2) C(2) < e^4 n^6 2^{1-n} = o(1).$$

Similarly, we can show that $q^*_{2,1}/q_1$ is $o(1)$. Now,

$$q_{2,0} \leqslant \sum_{i=3}^{n/2-1} \binom{n}{i} 2^{-i(n-i)} = O\left(n\binom{n}{3}2^{-3(n-3)}\right),$$

since the first term in this sum is the largest. We get $q_{2,0}/q_1 = o(1)$. With a similar argument we show that $q_{2,1}/q_1$ is $o(1)$. Applying formula (1) finishes the proof. □

References

[1] G.E. Andrews, The theory of partitions, in: Encyclopedia of Mathematics and its Applications 2 (Addison-Wesley, Reading, MA, 1976).
[2] B. Bollobás, Random Graphs (Academic Press, London, 1985).
[3] M.R. Brito, W.F. de la Vega, O. Meza and O. Ordaz, Viruses in graphs and digraphs, Rapport de Recherche 803, LRI, Université de Paris-Sud, Orsay (1992).
[4] G. Chartrand and L. Lesniak, Graphs and Digraphs (Wadsworth and Brooks / Cole, Monterey, CA, 2nd ed., 1986).